that says that knowledge comes from introspection of internal objects to avoid falling back into the claim that formal knowledge comes from acquaintance with concrete objects.

For the realist, reason alone must be the source of formal knowledge. Hence, the realist must try to develop an epistemology along the lines of the traditional rationalism of philosophers like Descartes and Leibniz, that is, one on which mathematical, logical, and linguistic knowledge is knowledge of *a priori* truths knowable on the basis of reason alone.

than nativism because the conceptualist is precluded from claiming that linguistic knowledge is *a priori* in the strong sense of "independent of experience" and from claiming that natural languages contain genuine necessary truths.

KATZ, JERROLD, Jr. REALISTIC RATIONALISM
CAMBRIDGE: MIT PRESS, 1998

Chapter 2

The Epistemic Challenge to Realism

2.1 Introduction

The epistemic challenge to realism in Benacerraf's ([1973] 1983) paper "Mathematical Truth" is to explain how spatiotemporal creatures like ourselves can have knowledge of objects with no spatiotemporal location. Although it seems clear that Benacerraf thinks that realism is unable to explain mathematical knowledge, his aim in this paper is not to refute realism. His aim is to make both sides in the controversy between realism and antirealism face up to their problems: doing justice to mathematical knowledge in the case of the realist and doing justice to mathematical truth in the case of the antirealist. Benacerraf's message is that no philosophy of mathematics, as it presently stands, is equal to the task of explaining both mathematical knowledge and mathematical truth.

In contrast, antirealists such as Gottlieb (1980, 11) and Field (1980, 98) claim that the perceptual inaccessibility of abstract objects refutes realism because it exposes realist epistemology as a form of mysticism. Most philosophers, though they are not prepared to dismiss realism outright, nonetheless find the circumstantial evidence of epistemological malfeasance sufficient to judge realism guilty until proven innocent. It is especially easy for them to reach this conclusion since their naturalist outlook inclines them to be suspicious of non-natural objects. All in all, few philosophers think that realism has much of a chance of meeting Benacerraf's epistemological challenge.

But arguments, not numbers, count in philosophy. In this chapter, I will try to show that Benacerraf's argument that mathematical knowledge is impossible if mathematical objects are beyond our causal reach, which underlies both the verdict that realism is guilty outright and the conclusion that it is guilty until proven innocent, is unsound. The argument rests on the false assumption that information from causal interaction with natural objects is a necessary feature of justification in *any* form of knowledge. The assumption is rarely questioned in contemporary philosophy because of the prevailing empiricist outlook and the absence of a rationalist epistemology that provides an alternative to experience as a basis for taking mathematical propositions to be true. But, since empiricism can be questioned and such a rationalist epistemology cannot be ruled out, the assumption is hardly invulnerable.

We saw in the introduction that many realists also accept the assumption. Restricting themselves to an acquaintance epistemology, they have ended up with an incoherent epistemology, or a position that is not realism, or empty hands. Realists, however, are in fact the last philosophers whose ontology requires them to explain mathematical knowledge on the basis of perceptual acquaintance. Since it is only a naturalist ontology that forces a philosopher to adopt the empiricist principle that all knowledge rests in part on causal interaction with natural objects, realists ought to be the first philosophers to reject the idea that evidence about natural objects is essential to knowledge about non-natural objects.

Because realists who have bought into an acquaintance epistemology are the majority of influential realists, and the rare influential realists such as Gödel who reject it have typically been misrepresented as accepting it, realism has yet to receive a defense against the charge of epistemological malfeasance. Since realism rejects all forms of naturalism, ontological, epistemological, and methodological, its explanation of our knowledge of pure mathematics and other formal sciences should be based on the rationalist's notion that the truths of pure mathematics and other formal sciences are truths of pure reason.

2.2 *Truth or Knowledge*

The issue is what is the best overall philosophy of mathematics and the formal sciences. An adequate philosophy of mathematics, Benacerraf ([1973] 1983) points out, has to provide both a plausible semantics for number-theoretic propositions and a plausible epistemology for mathematical knowledge. He argues that none of the available philosophies of mathematics can do this. Each satisfies one of these requirements at the expense of the other. "Separately," Benacerraf ([1973] 1983, 410) says, "[the requirements] are innocuous enough," but "jointly they seem to rule out almost every account of mathematical truth that has been proposed." I find Benacerraf's imposition of these requirements to be acceptable, and his assessment that no available philosophy of mathematics satisfies them to be entirely accurate.

As indicated, I reject the view that our best bet for obtaining an adequate philosophy of mathematics is further study of the concept of truth in mathematics. This view sees the principal obstacle to be either that some of the accounts that we have of mathematical truth are not fully or properly formulated or else that we do not as yet have all the accounts of mathematical truth. In contrast, I think that our best bet is further study of the concept of knowledge in mathematics. Insofar as the problem is the product of both a semantic and an epistemic requirement, the obstacle to an adequate philosophy of mathematics could be epistemological as well as semantic.

Benacerraf ([1973] 1983, 409) rightly requires that "an account of mathematical truth . . . must be consistent with the possibility of having mathematical knowledge," but that requirement, although weak enough to be generally acceptable, is not, as it stands, strong enough to rule out an epistemic solution. To do that, the requirement would have to be bolstered with some restriction of its notion of possibility to what the causal theory of knowledge allows as possible knowledge. Such a restriction is in place within the epistemology that Benacerraf thinks is broadly right. Referring to the "core intuition" of epistemologies like Goldman's (1967), Benacerraf ([1973] 1983, 413) claims that "some such view must be correct." This claim rules out an epistemic solution. The question is then what reason there is for thinking that such a causal epistemology is generally acceptable.

Benacerraf's ([1973] 1983, 413–14) reason for wanting an ontology that allows causal connection seems to be exclusively epistemic. If numbers are abstract objects, "then the connection between the truth conditions for the statements of number theory and any relevant events connected with people who are supposed to have mathematical knowledge cannot be made out," since our "four-dimensional space-time worm does not make the necessary (causal) contact with *the grounds of the truth of* [those statements]." This point, though undeniable, does not yet show that we cannot come to know abstract objects. It only shows that we cannot come to know them *in the way* we come to know concrete objects, that is, via a causal connection between ourselves and the objects of knowledge. To be sure, the condition that has to be satisfied to know number-theoretic statements cannot be one the satisfaction of which ensures that we are causally connected to the mathematical facts. But what grounds are there for thinking that the condition has to involve a causal relation to mathematical reality rather than some other epistemic relation?

The answer is empiricism. It is the empiricist principle that *all* knowledge depends on experience that sanctions generalizing the causal condition appropriate to cases of empirical knowledge such as Benacerraf's ([1973] 1983) case of Hermione's knowledge that "the black object she is holding is a truffle" to cases of mathematics and other types of formal knowledge. The price, however, is that now the acceptability of a causal condition on formal knowledge depends on the acceptability of empiricism. But, since empiricism hasn't been established, there is no argument that everyone has to accept the strengthened requirement that the justification condition for formal knowledge cannot be met without appealing to experience, and hence there are no uncontroversial grounds on which to invoke the strengthened requirement. Saying that causal contact with the objects of knowledge is a necessary condition for mathematical knowledge makes the requirement strong enough to rule out realism, but now the requirement is too strong to be generally acceptable.

Benacerraf's ([1973] 1983, 412–15) claim that the realist account of mathematical truth does not mesh with "our over-all account of knowledge" has no force against a realist account on which mathematical knowledge is purely *a priori*. Given what he means by "our over-all account of knowledge," his claim comes down to the assertion that a realist account of mathematical truth does not mesh with empiricism. But since it has yet to be shown that empiricism is the best overall theory of knowledge, the claim cannot be used to argue against a rationalist epistemology without begging the question. It is thus an open possibility that the condition for such knowledge is one that, if satisfied, ensures correspondence to mathematical facts, not, as it were, courtesy of our senses, but purely *a priori*, as rationalists have always thought.

Benacerraf recognizes this danger. In the unpublished manuscript from which much of the material in "Mathematical Truth" came, he (1968, 53) notes that the focus in that manuscript might also have been on the concept of knowledge. He indicates that the assumption that the source of the problem is our understanding of mathematical truth rather than our understanding of human knowledge reflects his personal confidence in a causal theory of knowledge, but he explicitly recognizes that arguing from that assumption might be criticized as "stacking the deck." Benacerraf (1968, 53) says that his

> . . . claim is that with the concepts of knowledge and truth, extricated as I have suggested, we do not seem to have adequate accounts of mathematical truth and mathematical knowledge. I am open to suggestion on how the analysis of either concept might be improved to remedy this defect.

We will proceed on the assumption that the obstacle to an adequate philosophy of mathematics can be located in the empiricist concept of knowledge.

2.3 *Toward an Epistemological Solution*

In this section, I want to show that, in addition to there being an option of seeking an improved concept of mathematical knowledge, there are prima facie reasons for thinking that an improved concept of knowledge is a better prospect than an improved concept of truth. On the one hand, Benacerraf's criticism of antirealism—the positions he ([1973] 1983, 406–7, 416) calls "combinatorial"—is a much stronger argument against antirealism than his formulation suggests, and, on the other hand, there are difficulties with the extension of the empiricist concept of knowledge to the formal sciences. (I will discuss Benacerraf's criticism at some length because it will have further applications later in the book.)

Benacerraf ([1973] 1983, 405–12) pointed out that realism has the advantage of allowing a uniform semantic treatment of mathematical and nonmathematical sentences, that is, one that treats the logical forms of mathematical sentences and corresponding nonmathematical sentences in a parallel way. On realism, (1) and (2) are both straightforward instances of (3),

(1) There are at least three perfect numbers greater than seventeen.

(2) There are at least three large cities older than New York.

(3) There are at least three FG's that bear R to a.

but, on the antirealist approaches he calls "combinatorial," (2) is not an instance of (3). Benacerraf ([1973] 1983, 410–12) bases his preference for a uniform semantics on the success of Tarskian semantics generally, the absence of an alternative semantics for combinatorial approaches, and the difficulty of coming up with an appropriate multiform semantics for such approaches.[1]

The difficulty of coming up with a multiform semantics might not be the worst of it for combinatorial approaches. The intuitive appeal of treating the semantic form of mathematical sentences like (1) and nonmathematical sentences like (2) in the same way suggests that we wouldn't want a multiform semantics even if we could come up with one. Burgess (1983, 1990, 7) made a start toward such a criticism in observing that the choice between a uniform and a multiform semantics for natural language belongs to "the pertinent specialist professionals"

1. This argument against antirealism must also contain an argument against a noncombinatorial nominalist view like Field's (1980, 1989) that takes reference to numbers to be reference to fictional entities and mathematical truth to be truth in a certain type of fiction. See the criticisms in section 2 of the previous chapter for such an argument.

in linguistics and suggesting that they would find a multiform treatment of sentences like (1) and (2) unacceptable. I think that Burgess is certainly right that the choice has to be made on the basis of which semantics best fits into the linguist's account of the grammar of natural language. He is also right, as I shall now argue, that a multiform treatment of sentences like (1) and (2) is unacceptable in linguistics.

If the intuition that (1) is as much an instance of (3) as (2) is correct, then what is wrong with treating (2) but not (1) as an instance of (3) is that sentences with essentially the same grammatical structure are treated as essentially different in grammatical structure. To be sure, there are cases in linguistics where it might seem that sentences with the same grammatical structure are treated as different in grammatical structure. A classic example of such a case in the history of generative linguistics is the postulation of different underlying structures for (4) and (5).

(4) John is easy to please.

(5) John is eager to please.

However, the linguist's treatment of such cases differs from the combinatorialist's treatment of cases like (1) and (2). The difference is that cases like (4) and (5) are ones where the appearance of sameness of grammatical structure does not go below the surface: "John" is the direct object of the infinitive in (4) and the subject of the infinitive in (5). The linguist's desire to account for those facts—in accord with the prevailing theory of grammatical relations (see Chomsky 1965)—motivated a transformational analysis of (4) and (5) on which surface similarity masks deep grammatical difference. Hence, the linguist's treatment sacrifices no more than a surface grammatical similarity, and the sacrifice is for the higher good of preserving a deeper grammatical truth. Since the different grammatical treatment of sentences like (4) and (5) is *grammatically* driven, such sentences are the "exceptions" that prove the rule that grammatically similar sentences are to be similarly described.

The transformational analysis of (4) and (5) is both appropriate to linguistics and properly implemented. The linguist's aim is to obtain a more encompassing analysis of grammatical structure than is possible on a description that preserves surface similarity. The analysis is properly implemented because the assignment of different underlying syntactic structures is based on grammatical facts about the sentences. In contrast, the multiform analysis of (1) and (2) is neither appropriate to linguistics nor properly implemented. The combinatorialist's aim is the linguistically irrelevant one of trying to avoid a commitment to realism

in the foundations of mathematics. The implementation of the analysis is improper not only because there is no strong grammatical intuition reflecting a difference in their grammatical structure, but also because there is not even a hint of the kind of underlying grammatical differences we find in connection with cases like (4) and (5). From a grammatical perspective, (1) and (2) are each as much an instance of (3) as the other.

(1) is not an isolated case. There are infinitely many sentences of English in which number terms occur in a referential position. Hence, if the concerns of a partisan viewpoint in the philosophy of mathematics are allowed to decide questions of grammatical structure, distinctions reflecting no grammatical differences will be made over a wide segment of the language. Since such distinctions are only philosophically motivated, a multiform semantics would compromise the autonomy of linguistics. Linguistic argumentation would degenerate into philosophical debate—Why should the linguist let the concerns of antirealism decide? Why not the concerns of realism? And so on. To preserve the autonomy of linguistics and the integrity of its argumentation, only linguistic considerations can be allowed to determine the description of sentences.[2]

Related to the foregoing semantic doubts about the extension of the empiricist concept of knowledge to the formal sciences are doubts about whether there is anything in the natural world to which knowledge about numbers, sets, propositions, and sentences can be causally connected. Not only is there, as it were, no truffle, there doesn't seem to be anything to which such knowledge can be causally related, not even in the remote way in which theoretical truths in physics can be causally related to events in cloud chambers or pictures from radio telescopes. There seem to be no natural objects to serve as the referents of the terms in (6)–(8) and no natural facts to which such truths may be taken to correspond.

(6) Seventeen is a prime number.

(7) No proposition is both true and false.

2. Someone might reply that it would be a good thing if the present disciplinary boundaries were to disappear and if questions within disciplines could be decided partly on the basis of arguments from other disciplines in a fully interdisciplinary way. It is not clear to me what would be so good about this, either for scientific disciplines or for philosophy. If decisions within disciplines became so radically interdisciplinary, it would wreak havoc with argumentation about the nature of phenomena in a scientific discipline, since its practitioners could then substitute philosophical arguments for scientific considerations. We would forfeit the constraint to save the phenomena. Nor would it be good for philosophy to be without an independent scientific characterization of knowledge by which to judge philosophical accounts of knowledge.

(8) An anagram is an expression that is a transposition of the letters of another expression.

Other doubts about whether the causal condition generalizes stem from the argument of rationalists such as Arnauld, Leibniz, and Kant that experience cannot provide knowledge of the fact that mathematical and logical truths couldn't be otherwise. Quine and other naturalists have denied that there are necessary truths, but, in the fullness of time and modality, those arguments have not proven, as, for example, Marcus (1990) has argued, to have the force they were once thought to have.[3] Given that there is nothing in the natural world that can explain the necessity of such truths, they are at least prima facie counterexamples to a causal condition on all formal knowledge.

Benacerraf quite correctly claims that a realist account of mathematical truth does not mesh with an empiricist account of knowledge. From the perspective of considerations such as those above, this can be taken to mean only that we cannot know truths about mathematical objects in the same way we know truths about natural objects. Furthermore, from that perspective, the existence of mathematical knowledge shows that there must be a different way of knowing mathematical truths.

2.4 *Mysticism and Mystery*

Some antirealists would dismiss the possibility of an epistemology for abstract objects. They take the view that any epistemology for any form of realism is mysticism. Gottlieb (1980, 11) says, "*Abstract entities are mysterious and must be avoided at all costs.*" Field (1989, 59) says that the realist "is going to have to postulate some *aphysical connection*, some *mysterious grasping*." Chihara (1982, 215) says that the realist's appeal to Gödelian intuition is "like appealing to experiences vaguely described as 'mystical experiences' to justify belief in the existence of God." Dummett (1978, 202) says that Gödelian intuition "has the ring of philosophical superstition." These antirealists think such things because they think that, in the absence of objects acting directly or indirectly on us, abstract objects might be any way whatsoever for all we would know. Hence, they think that the realist's claim that we know how things are in the realm of abstract objects in spite of having no natural connection to it can only be based on the pretense that we have a supernatural connection.

There are two things wrong with this antirealist view. One is the unwarranted assumption of an empiricist epistemology on which we have already commented and on which we will have more to say in the following sections. The other is a confusion between mysticism and mystery. No one would deny that there is a mystery about how we can have knowledge of abstract objects. But such extreme antirealists make far too much of that particular philosophical mystery. Philosophy is full of such mysteries. Every philosophical problem is one. Each reflects something we find incomprehensible about the world or our knowledge of it. In the case at hand, we find truth in mathematics and other formal sciences incomprehensible. But the obscurity of the rational mechanism is no grounds for dismissing the claim that pure reason does what, from the common-sense standpoint, it appears to do. Why should the realist's project of explaining how spatiotemporal creatures like ourselves obtain their knowledge of causally inert objects deserve more disparagement than the naturalist's project of explaining the equally mysterious process by which we obtain conscious experience from the physical effects of material objects or the empiricist's project of explaining why it is rational to believe that the future will be like the past? To be sure, some realists have strayed off into mysticism, but it is just guilt by association to criticize all realists for the sins of some.

A philosophical mystery is no grounds for crying "mysticism." Mysticism involves the claim to have a means of attaining knowledge beyond our natural cognitive faculties. Those who cry "mysticism," "superstition," and the like perhaps need to be reminded of the fact that our sensory faculties do not exhaust those faculties. Sophisticated empiricists recognize an autonomous rational faculty as essential for knowledge. For example, Benacerraf's ([1973] 1983, 413) statement that "knowledge of general laws and theories, and, through them, knowledge of the future and much of the past" is "based on inferences based on [perceptual knowledge of medium-sized objects]" seems to recognize that the inferential operations of reason can't be reduced to the operations of our senses. Such empiricists do not think of reason as a device for cataloguing data based on similarity comparisons of sensory information, but as an inferential engine that, together with sense experience, is necessary for the justification of our beliefs about the

3. Given that the argument for the causal condition in the case of mathematical knowledge is so weak, it seems that the prevalence of the naturalistic outlook is the only thing that instills confidence in the prospects of empiricism as a general theory of knowledge. Naturalism supplies the premise required to generalize from a causal condition on justification in the uncontentious case of empirical knowledge to one on justification in all cases of knowledge. Having an ontology that says that all objects of knowledge are uniformly natural objects and also accepting a causal condition on knowledge of natural objects exerts overwhelming pressure to generalize the causal condition to knowledge of numbers, sets, propositions, sentences, meanings, and the like. Katz (1990b) argues that the principal arguments that twentieth-century philosophers have given for naturalism are deeply flawed.

world. The bottom line is that the epistemology for mathematical knowledge to be set out in the next section, which is based on our natural cognitive faculty of reason, should dispel the charge of mysticism. Antirealists will no doubt be less than happy about this epistemology, but they will not be able to dismiss it as mysticism.

2.5 *An Outline of a Rationalist Epistemology*

Our project is intended to provide an "improved" concept of knowledge, improved in the sense that it overcomes the principal fault of traditional realism: its lack of a plausible epistemology. As we observed in the last chapter, Gödel ([1947] 1983, 484) points us in the right direction. He makes it clear that the "mathematical intuition"—we may say, the rational faculty—on which mathematical knowledge rests "cannot be associated with actions of certain things upon our sense organs" or with "something purely subjective, as Kant asserted." Gödel ([1947] 1983, 484) went on to say that its presence in us "may be due to another kind of relationship between us and reality." These remarks focus the task for developing a realist epistemology that can meet the epistemological challenge. The task is to provide an account of this other kind of relationship that explains how we come to stand in that relationship to the realm of abstract objects and, with no window on that realm, come to know what things are like there.

2.5.1 *Epistemic Conditions*

Like Benacerraf ([1973] 1983, 414), I will assume for the sake of argument that knowledge is justified true belief. Belief, truth, and justification will not be understood in any special philosophical sense; rather, they will be taken in a sense as close to their familiar sense as possible. Someone has a belief about something when he or she takes a proposition about it to be true. A proposition is about something when one of its referring terms refers to it. A proposition is true when the things it is about satisfy its truth condition, that is, when the facts are as the proposition says they are. Our belief about something is justified when we have adequate grounds for taking the proposition about it to be true. These characterizations fit the epistemic assumptions of Benacerraf's statement of the epistemic challenge and are compatible with a wide range of views in the theory of knowledge.

2.5.2 *The Belief Condition*

No aspect of the realist's epistemology can entail causal contact with abstract objects. But the constraint that mathematical and other formal knowledge be based on reason alone, though it applies to the truth and

justification conditions, does not apply to the belief condition. In the case of the belief condition, the states that constitute the taking of a proposition to be true can derive features of their ideational content from experiential relations to natural objects, since abstract objects are not involved in such relations.[4]

The possibility that the ideational content of doxastic states depends in some respects on experience shows that nativism does not have to be a component of the rationalist position. Nativism is a theory about the acquisition of the concepts with which our cognitive faculties work. It can be understood as the claim that the concepts required to form beliefs about (*inter alia*) abstract objects are either themselves inherent constituents of our cognitive faculties or else derivable from concepts that are inherent constituents on the basis of principles that also belong to those faculties. (See Katz 1979; 1981, 192–220.)

The most influential contemporary form of nativism is Chomsky's (1965, 47–59). His nativism hypothesizes that the child has innate knowledge of the grammatical structure of natural language, and also innate principles for putting this knowledge to use in acquiring (tacit) knowledge of a natural language (i.e., competence). Given that the child's innate knowledge represents the full range of possible competence systems for natural languages, the child's task is to choose, on the basis of a sample of utterances of a language, the system in this range that will give it fluency in the language. There are various specific hypotheses about the way in which the child chooses, such as parameter setting, hypothesis testing, and so on, but the nativist's general claim is that the child's innate grammatical structure is so rich that experience does no more than determine which competence system is the right one for the sample of utterances to which it is exposed.

Now a philosopher might well think, as I do, that both nativism and rationalism are correct. One might think that the former is correct because some such theory as Chomsky's accounts best for the facts concerning the child's acquisition of a competence system, and one might think that the latter is correct because some such theory as the one I will describe best accounts for the epistemological and semantic facts about knowledge in mathematics and other formal sciences. But equations of nativism and rationalism, such as Chomsky's (1965, 1966), conflate psychological questions with epistemological questions. Chomsky's conflation doubtless stems from the prior conflation of psychology and epistemology that comes with conceptualism.

4. This discussion is a revision of the view about the belief condition in Katz (1995, 500–502). It was prompted by a query from Glenn Branch.

2.5.3 *The Truth Condition*

The correspondence of the mathematical proposition and the mathematical fact in virtue of which the proposition is true involves no contact between an abstract and a concrete object because, on our realism, such correspondence holds among abstract objects. On the one hand, mathematical facts are facts about abstract objects. On the other hand, both of the principal conceptions of propositions, the Fregean and the Russellian, enable us to construe number-theoretic and other formal propositions as abstract objects. On the Fregean conception, propositions are senses of sentences, and it is quite natural for the mathematical realist to say, in accord with linguistic realism, that senses of sentences are abstract objects. On the Russellian conception of propositions, a proposition is an ordered n-tuple consisting of the object(s) the proposition is about and the property or relation the proposition ascribes to it (them). Here too mathematical propositions can be taken as abstract objects because, for us, all of the components of such propositions—the mathematical objects in the n-tuple as well as the properties and relations—are abstract objects. Since both mathematical propositions and the facts they are about are abstract, mathematical truth is simply an abstract relation between abstract objects.

2.5.4 *The Justification Condition*

On our epistemology, what counts as adequate grounds for the truth of a proposition depends on the nature of the proposition, which, in turn, depends on the nature of the objects the proposition is about. So, propositions about natural objects are empirical and propositions about abstract objects are nonempirical. We can take something like Benacerraf's ([1973] 1983, 413) concept of empirical knowledge as specifying what counts as adequate grounds for the truth of propositions about natural objects. We require a concept of *a priori* knowledge to specify what counts as adequate grounds for the truth of propositions about abstract objects.

In *Language and Other Abstract Objects*, I (1981, 200–216) made a suggestion about how to understand *a priori* knowledge of abstract objects. It involved two thoughts. The first was that the entire idea that our knowledge of abstract objects might be based on perceptual contact is misguided, since, even if we had contact with abstract objects, the information we could obtain from such contact wouldn't help us in trying to justify our beliefs about them. The epistemological function of perceptual contact is to provide information about which possibilities are actualities. Perceptual contact thus has a point in the case of empirical propositions. Because natural objects can be otherwise than they actually are (*non obstante* their essential properties), contact is

necessary in order to discover how they actually are. In some possible worlds, gorillas like bananas, while in others they don't. Hence, an information channel to actual gorillas is needed in order for us to discover their taste in fruit. Not so with abstract objects. They couldn't be otherwise than they are. They have all their intrinsic properties and relations necessarily.[5] Purely abstract properties and relations of abstract objects cannot differ from one world to another. The way abstract objects actually are with respect to their intrinsic properties and relations is the way they must be. Hence there is no question of which mathematical possibilities are actualities. Unlike what is actually the case about gorillas, what is actually the case about numbers is what *must* be the case about them. In virtue of being a perfect number, six must be a perfect number; in virtue of being the only even prime, two must be the only even prime. Since the epistemic role of contact is to provide us with the information needed to select among the different ways something might be, and since perceptual contact cannot provide information about how something must be, contact has no point in relation to abstract objects. It cannot ground beliefs about them.

In *On the Plurality of Worlds*, Lewis (1986b, 111–12) expresses a similar thought. He says that the necessity of a mathematical proposition exempts it from the requirement on empirical propositions to show that they counterfactually depend on the facts to which they correspond. According to Lewis (1989b, 111), counterfactual dependency is not required for mathematical propositions because "nothing can depend counterfactually on non-contingent matters. For instance, nothing can depend counterfactually on what mathematical objects there are. . . . *Nothing sensible can be said about how our opinions would be different if there were no number seventeen*." (Italics mine)

Field (1989, 237) rightly objects that we can sensibly say how things would be different if the axiom of choice were false. Furthermore, if what Lewis says were so, there would be no reductio disproofs of necessarily false statements in the formal sciences, since such proofs

5. My claim is that all intrinsic or formal properties and relations of abstract objects are necessary. Sometimes it is held that some properties and relations of abstract objects are contingent, e.g., the relation I bear to the number seventeen when I am thinking of it. On my view, this is not the case (see chapter 5, section 3). A somewhat trickier case: a word has the property of being coined at a particular time (but might have been coined at another time). Since I want to treat the words of a language as types in Peirce's sense, and hence as abstract objects, coining a word is not creating a word of the language. As I (1981, chapter 5) argue elsewhere, what happens when a word is coined is that speakers of a language begin to use tokens of the word as tokens of that type under conditions that lead to a change in their competence. Clearly, a word type's having (or not having) a representation in the competence of speakers is not one of its intrinsic or formal properties and relations.

begin with the supposition that a necessarily false statement is true, which, on Lewis's view, is "nothing sensible." For example, in a reductio proof that the square root of two is irrational, we suppose counterfactually that (9) is true

(9) There is a rational number equal to the square root of two.

and then, reasoning from this supposition, we go on to spell out how things would be different, given the existence of such a rational number. We say such things as that there would be numbers that are both even and odd. The absurdity of a reductio is precisely what it *is* sensible to say about how things would be different if the necessarily false supposition were true.

At one time, Lewis (1973, 24–26) held the opposite view. Presumably, the change took place because of the difficulty of reconciling the earlier view that something sensible can be said about how our opinions would be different if a necessarily false proposition were true with Lewis's possible worlds conception of propositions.[6] Insofar as we have no commitment to the possible worlds conception of propositions, we have no reason to stick to it at the prohibitive cost of not being able to make sense of reductio proofs.[7]

To express the otiosity of contact in the case of formal knowledge unproblematically, we require an alternative conception of propositions on which necessarily false sentences express propositions. One such conception is that propositions are senses of sentences. (See Katz 1972, 1977; Smith and Katz in preparation.) With this conception, we can specify the condition for supposability as the meaningfulness of the clausal complement of the verb "suppose." A sentence expresses no supposition when and only when, like in (10),

(10) Suppose that seventeen loves its mother.

the clausal complement of "suppose" has no sense, and hence provides no object for the propositional attitude. When the complement has a sense, *even* one that is necessarily false, like the complement in (11),

(11) Suppose that some propositions are both true and false.

there is an object for the propositional attitude, namely the sense of the clause, and the whole sentence expresses a supposition. Thus, on our alternative conception of propositions, there is something to be supposed in the case of necessarily false sentences, and hence we can make sense of reductio proofs.

Given this condition for supposability, reductio proofs in the formal sciences are tests of necessary truth based on an exploration of the logical consequences of supposing a necessary truth to be false. In such proofs, we first suppose that what a necessary truth asserts is not the case. Then, by deriving an explicit inconsistency, we expose the fact that the proposition we supposed is a contradiction, from which fact we then infer that the denial of the proposition is true. Hence, something sensible can be said about how things are on a necessarily false supposition, namely, that things are every way they can supposably be.[8]

In expanded form, this is the first of the two thoughts about knowledge of abstract objects in *Language and Other Abstract Objects* (Katz 1981). The second thought was that it is no loss for the epistemic function of contact not to carry over to the realm of abstract objects because our reason is an appropriate instrument for determining how things must be in that realm. It is a truism of mathematical practice that reasoning can show that mathematical objects could not be otherwise than as mathematics presents them. When mathematicians proved the proposition that the square root of two is irrational, they showed, on the basis of reason alone, that its truth conditions are satisfied no matter what one supposed about the numbers. But, if the light of reason enables the mind's eye to see that the square root of two is

6. In possible worlds semantics, propositions are sets of possible worlds: the proposition *P* is the set of possible worlds in which *P* is true. Either contradictions are true in the null set of possible worlds or they express no proposition. But, if we say that they are true in the null set of possible worlds, it does not seem possible to maintain the distinction in Lewis's earlier position between its being sensible to say "*p!* + 1 is prime" and "*p!* + 1 is composite" on the supposition (1) "there is a largest prime *p!*", but not on "there are six regular solids" or "pigs have wings." Since the former are taken to be sensible things to say, it would also have to be sensible to say "pigs have wings" on the supposition (ii) "pigs have wings and are wingless." But, since *ex hypothesi* a contradiction is true in exactly the null set of possible worlds, all contradictions express the same proposition. Since (i) and (ii) express the same proposition, if it is sensible to say "*p!* + 1 is prime" and "*p!* + 1 is composite," it must also be sensible to say "pigs have wings." So, it seems that, to avoid having to concede that the latter is sensible, Lewis decided to say that neither is. (See Katz 1996a for a discussion of the shortcomings of possible worlds semantics.)

7. Even if reductio proofs are not necessary because a direct proof can be given for each theorem established by a reductio proof, the cost of not being able to make sense of such proofs is still prohibitive. The existence of direct proofs doesn't change the fact that reductio proofs are proofs.

8. Moreover, since, on that notion, necessarily false sentences with different senses can express different propositions, separate reductio proofs are to be kept separate. We have to keep them separate to be able to say that things are the same on the supposition that there is no number seventeen as they are on the different supposition that there is no number two or that there is a largest prime.

irrational, of what relevance can it be that the body's eye cannot see numbers?

Reductio proofs provide the most straightforward way of showing that a mathematical truth is a necessary truth. This is because such proofs (explicitly) begin with the supposition that the proposition is false and go on to exclude every possibility of the supposition's being true. They show that there is only one possibility of how the mathematical objects in question might be because other putative "possibilities" are impossibilities. The following argument (A) that two is the only even prime is an example.

> (A) We see that two is an even prime. Supposing that another number is even and prime, that number must be either less than two or greater than two. If the number is less than two, it has to be one, but then it is not even. But, if the number is greater than two, then, since it is even, it is divisible by two, and hence not prime. Since the law of trichotomy cannot fail here, there is no even prime other than the number two.

A proof provides us with adequate grounds for knowledge of a proposition about abstract objects by showing that it is impossible for the objects to be other than as the proposition says they are. But the question arises of how a proof establishes the necessity of its conclusion when, typically, the conclusion is simply a statement that the mathematical objects in question have a certain property rather than a statement expressing a modal predication about how they have the property. For example, the conclusion of (A) simply states that every number other than two lacks the property of being even and prime. It is not itself a statement that they necessarily lack the property. So, how can (A) establish that its conclusion is necessary?

The answer is that a proof of a proposition *P* that is not itself a modal statement establishes the modal statement "Necessarily, *P*" in virtue of the fact that it is a proof of *P*. The essence of proof consists in reasoning so close-textured, so tight, that it excludes every possibility of the conclusion's being false. Mathematicians sometimes say that there are no "holes" or "gaps" in a proof. This is not just a matter of logical structure. The tightness of a proof derives not only from the absence of any counterexample to any of the steps from premises to conclusion, but also from the absence of any counterexample to any of the premises. The argument (A) is a proof that two is the only even prime not only because none of its inferential steps can be faulted, but also because its premises cannot be faulted either. Hence, there is no possibility of two not being an even prime. Thus, (A) shows that two is necessarily the only even prime, not in virtue of a modal conclusion, but in virtue of reasoning so tight that *every* possibility of another even prime is ruled out.[9]

2.6 The Order of Knowledge

There are two respects in which the foregoing account of knowledge falls short of meeting Benacerraf's epistemic challenge to realism. First, it is only an explication of some of the rational methods for acquiring formal knowledge, and hence we have yet to establish that none of the remaining methods presupposes contact with abstract objects. Second, the account does not explain how a full rationalist epistemology would meet the epistemic challenge. The present section completes our sketch of the rationalist epistemology. The next section explains how the challenge is met.

2.6.1

The aspects of reason remaining to be discussed are those that are responsible for the steps from knowledge of simple mathematical facts to knowledge of mathematical laws and theories. Since causal contact plays no role in explaining knowledge of the simple mathematical facts that underlie knowledge of mathematical laws and theories, dependency of these aspects of reason on contact with abstract objects seems unlikely on the face of it. Nevertheless, to be sure there is no depend-

9. Tightness is different from the properties of informativeness and depth. A completely tight proof may be less informative about what is going on mathematically—may give less insight into the mathematical structure—than a proof with a hole. Further, tightness and informativeness are both different from depth, what the proof tells us about the bigger mathematical picture. Truth table proofs of tautologies and solutions to chess problems are completely tight and completely informative, but they are not mathematically deep. Thus, a proof with a hole in it can be mathematically significant if it is informative or deep, but, of course, for the purpose of having mathematical beliefs on which we can rely absolutely, an uninformative or shallow proof is as good as an informative or deep one.

It is widely held that knowledge is reliable in the sense of being theoretically and practically dependable. In empirical knowledge, reliability is typically explained on the grounds that the knowledge rests on evidence from either direct or indirect causal contact with the natural objects it is about. Since the process of arriving at empirical beliefs monitors those objects, empirical investigation provides grounds for confidence in the evidence, and hence the beliefs it supports. In the case of mathematical knowledge, we can explain reliability in terms of tightness of proof. The tightness of mathematical proofs underwrites our confidence that their conclusions represent the numbers as they are. Every possibility of the numbers being otherwise has been excluded because there are no gaps.

ency, we need to see how a rationalist epistemology handles the ascent from knowledge of basic mathematical facts to knowledge of mathematical laws and theories.

The ascent from basic facts to knowledge of laws and theories is a feature of both *a priori* and *a posteriori* knowledge. Recognizing that there are theoretical cases of empirical knowledge in addition to basic cases like Hermione's knowledge that what she is holding is a truffle, Benacerraf ([1973] 1983, 413) qualifies his causal condition as "an account of our knowledge about medium-sized objects, in the present" and goes on to say:

> Other cases of knowledge can be explained as being based on inference based on cases such as these. . . . This is meant to include our knowledge of general laws and theories, and, through them, our knowledge of the future and much of the past.

This is, in effect, to introduce an order of knowledge into empiricist epistemology: there is *basic knowledge* of "medium-sized objects, in the present" and *transcendent knowledge* of "general laws and theories, and, through them, . . . knowledge of the future and much of the past."

Since there are general laws and theories in the formal sciences, a rationalist epistemology will have to describe a similar order of knowledge in the formal sciences. Just as Benacerraf's empiricist epistemology posits basic, observational knowledge of properties of medium-sized objects, our rationalist epistemology correspondingly posits basic ratiocinative knowledge of evident properties of abstract objects, e.g., the knowledge that four is composite. Just as Benacerraf's empiricist epistemology posits transcendent knowledge of empirical laws and theories, a rationalist epistemology correspondingly posits transcendent knowledge of formal laws and theories.

A sharp observational/theoretical distinction in the natural sciences has proven notoriously difficult to draw, so much so that many philosophers of science have given up trying to draw it. A sharp basic/transcendent distinction in the formal sciences, though, as far as I know, little investigated, is unlikely to prove more tractable. But, in the present investigation, it is no more necessary to try to draw such a distinction for the formal sciences than it was for Benacerraf to draw one for the natural sciences. We are no more concerned with describing science than he was.

Nonetheless, we can sketch a rough distinction. In the case of the natural sciences, for the most basic of basic knowledge, we have the case of seeing the color of the litmus paper with our own eyes. As we move away from this extreme to less basic knowledge, we have cases of observation that depend more and more on such artificial devices

as electron microscopes, radio telescopes, and the like that boost the power of our natural faculties. Since our confidence in such devices depends on laws and theories, theoretical considerations play a role in connecting what we directly perceive with what we count as observation. As we move from basic knowledge to transcendent knowledge, we have a shift in focus to establishing general laws and theories. Observational methods still play a role, but systematic considerations increasingly dominate.

The distinction in the formal sciences is similar. At the extreme of basic knowledge, we have the case of seeing—though not with our eyes—that four is composite. In other cases of basic knowledge, such as our knowledge that two is the only even prime, our insight is extended on the basis of reasoning and the use of computational devices that boost the power of our natural ratiocinative faculties. Since our confidence in such devices depends on laws and theories, theoretical considerations play a role. As we move from basic knowledge to transcendent knowledge, we have a shift in focus to establishing general laws and theories.[10] Insight still plays a role, but systematic considerations increasingly dominate.

In the formal sciences, it is common to refer to seeing that something is the case as "intuition" and to take such immediate apprehension as a source of basic mathematical knowledge.[11] Mention of intuition raises two immediate concerns. One is the cry of "mysticism" on the part of some radical antirealists. In this connection, the comments earlier in this chapter on the difference between mysticism and mystery apply. In addition to those comments, I will quickly discuss Wittgenstein's criticisms of intuition. Wittgenstein (1953, sec. 213) dismissed intuition as a source of knowledge, referring to it as "an unnecessary shuffle." He had two reasons for thus dismissing it. First, he thought that nothing is gained by invoking intuition because the signs that make up the speech of the "inner voice" of intuition would require interpretation just as much as the signs that make up the speech of the public language. In *The Metaphysics of Meaning,* I (1990b, 159–61) argued that

10. There is a story about a famous mathematics professor (different versions of the story mention different professors) who introduced a lemma as intuitively obvious in the course of proving a theorem. When a student didn't see that the lemma is intuitively obvious, the professor retired to examine the lemma, and, after an hour, returned to announce, "I was right; it is intuitively obvious."

11. The emphasis on the common notion of intuition should not obscure cases of basic mathematical knowledge that do not depend on rational operations encompassed within a single grasp of structure (though they might be thought of as concatenated intuitions). Examples are the theoretically unmediated inferences underlying our knowledge that two is the only even prime and our knowledge that a cube has twelve edges.

Wittgenstein's inner voice characterization of intuition is no more than a caricature of intuition in mathematics, logic, and linguistics, and I (1990b, 135–62) showed that within a realist framework there is no problem about a regress of interpretations. This solution is essentially a version of the solution to Kripke's puzzle about rule following that I shall propose in chapter 4.

Wittgenstein's other reason is based on the observation that intuition is not always reliable: "if it can guide me right, . . . it can also guide me wrong." Wittgenstein concludes that it is a mistake to appeal to intuition to pin down the interpretation of an initial segment of a series so that we can go on in the right way. This criticism, insofar as it goes beyond the previous one, is simply a misrepresentation of the commitment that comes with accepting intuition as a source of knowledge of basic mathematical and logical truths. There is no strong commitment to the infallibility of intuition. To be sure, intuition does not always do its job properly. Intuitive "seeing" is sometimes not reliable, but in this respect it is no different from visual seeing. Since we don't dismiss sight as a legitimate source of basic knowledge because there are times when our eyes deceive us, it is hard to see why we should dismiss insight as a legitimate source of basic knowledge because there are times when it deceives us. Like judgments based on sight, those based on insight can be checked against similar judgments and against theories based on them. Once intuition is integrated into a systematic methodology that enables us to correct unclear and deceptive cases on the basis of a broad range of clear cases and principles derived from them, Wittgenstein's worry that intuition sometimes gives the wrong guidance disappears.

The other concern is to avoid a possible misunderstanding of the notion of intuition. The notion of intuition that is relevant to our rationalist epistemology is that of an immediate, i.e., noninferential, purely rational apprehension of the structure of an abstract object, that is, an apprehension that involves absolutely no connection to anything concrete. The misunderstanding that I want to avoid is a confusion of this notion of intuition with a Kantian notion. The danger is real, since the most serious attention that the notion of intuition has received in recent literature in the philosophy of mathematics has been Parsons's (1980) development of a Kantian concept of intuition that involves a connection to something concrete in sense perception. I want to make it clear that Parsons's concept is not what I mean, because that connection prevents it from playing any role in a strictly rationalist epistemology for pure mathematics or any other pure formal science.

This is not to say that Parsons's concept isn't relevant to our approach to the foundations of the formal sciences. It may be useful in connection with what I call composite objects in chapter 5, particularly those that involve the type/token relation, such as drawn geometric figures and linguistic utterances (and inscriptions). I believe that the discussion in that chapter, particularly the distinctions drawn between composite objects and abstract objects and between the application of pure arithmetic and the application of pure geometry, is essential to seeing Parsons's work in the right light.

It is also crucial to the notion of intuition in our sense that intuitions are apprehensions of structure that can reveal the limits of possibility with respect to the abstract objects having the structure. Intuitions are of structure, and the structure we apprehend shows us that objects with that structure cannot be certain ways. Consider some examples. The intuition of the number four as a composite of two and two shows the impossibility of four's being a prime number. The intuition of the logical structure of an instance of modus ponens shows the impossibility of the truth of the premises without the truth of the conclusion. The intuition of the grammatical structure of "I saw the uncle of John and Mary" shows the impossibility of the sentence's having just one sense. As Descartes pointed out, when intuition is clear and distinct, as it is in such cases, we "could have no occasion to doubt it." What is present to our minds in a clear and distinct intuition of abstract objects is the fact that their structure puts the supposition of their being otherwise than as we grasp them to be beyond the limits of possibility.

The rationale for claiming that it is intuition that is the source of basic mathematical and other formal knowledge is something like the precept that Holmes recommends to Watson in *The Sign of Four.* Holmes says, "How often have I said to you that when you have eliminated the impossible, whatever remains, *however improbable,* is the truth?" There are cases in which we can eliminate everything but intuition as a possible explanation of how it is known that a premise or a step in a proof has no counterexample. In cases like the compositeness of four, the pigeon-hole principle, the indiscernibility of identicals, and the ambiguity of "I saw the uncle of John and Mary" or the well-formedness of "The cat is on the mat," there is no explanation other than intuition for the fact that ordinary, unsophisticated people, without expert help, immediately grasp the truth.

Consider the pigeon-hole principle. Even mathematically naive people immediately see that, if m things are put into n pigeon-holes, then, when m is greater than n, some hole must contain more than one thing. We can eliminate prior acquaintance with the proof of the pigeon-hole principle, instantaneous discovery of the proof, lucky guesses, and so on as "impossibilities." The only remaining explanation for the immediate knowledge of the principle is intuition.

Rationalists have sometimes extended this Holmesian defense of intuition, arguing that our knowledge of logic itself rests *au fond* on the exercise of intuition. Ewing (1947, 26) points out that an inferential step

> . . . must either be seen immediately or require further argument. If it is seen immediately, it is a case of intuition; if it has to be established by a further argument, this means that another term, D, must be interpolated between A and B such that A entails D and D entails B . . . , but then the same question arises about A entailing D, so that sooner or later we must come to something which we see intuitively to be true, as the process of interpolation cannot go on *ad infinitum*.

If the critics of intuition want to challenge the claim that it is a source of basic formal knowledge, they owe us an alternative explanation of our knowledge in cases of the kind that rationalists explain in terms of intuition.

Intuition, like empirical observation, has both a horizontal and a vertical limit, which make a transcendent mode of knowledge necessary: on the one hand, there are too many fundamental objects in the domain, e.g., natural numbers, and, on the other, there are principles that express relations among the objects of intuition that are too general to be apprehended in those objects, e.g., mathematical induction. Transcendent formal knowledge, like transcendent empirical knowledge, is based on inferences based on basic knowledge. Such inferences generalize basic knowledge and bring basic and transcendent knowledge into an integrated, coherent, total system under the guidance of an ideal of theoretical systematization. The ideal is best thought of as part of the general concept of knowledge rather than as a special feature of the concepts of empirical and formal knowledge. Thus, many aspects of theoretical knowledge in the natural and formal sciences—for example, the simplicity of theoretical systems—are imposed, as it were, from above.

In the formal sciences as well as in the natural sciences, theoretical considerations can be brought to bear in the process of systematization to determine the status of intuitively unclear cases and to correct intuitive or theoretical judgments that have been mistakenly accepted. As suggested already, systematization compensates for the fallibility of the processes of acquiring basic and transcendent knowledge by bringing considerations from one part of the overall system to bear on issues in another. Thus, the ideal of systematization gives a holistic character to justification in the formal and empirical sciences, but this is an epistemic holism that has nothing to do with a semantic holism such

as Quine's (1961c, 43). Our holism does not claim that "the whole of science" is the smallest independently meaningful linguistic unit. Rather, assuming a prior correlation of sentences and senses in natural languages, our holism concerns the ways in which propositions in a particular system in a formal science obtain their support from one another and from the basic knowledge on which the theory rests.

It is clear, according to our rationalist epistemology, that formal knowledge is not an exclusively bottom-up affair in which all basic knowledge is established prior to and independently of transcendental knowledge and systematization. Exactly how much foundationalism there is can be left open here, just as Benacerraf left the parallel question open in his account of empiricist epistemology. The extent to which transcendent knowledge must be anchored in basic knowledge can be treated as a more general question about reason's overall ideal of systematization, and, as such, the question is independent of the issues here. Thus, our methodological holism is compatible with various forms and degrees of foundationalism.

2.6.2

The holistic character of systems of formal knowledge allows for the possibility that not everything counted as knowledge at the level of transcendent knowledge can be shown to be necessary. There is, as far as I can see, no way to show that the justification of formal knowledge is uniformly a matter of excluding every possibility of falsehood. This does not mean that only principles for which we can exclude every possibility of falsehood count as knowledge in the formal sciences. Some principles at the transcendent level might count as knowledge, even though we cannot show that there is no possibility of their falsehood, because we can show that there is no possibility of achieving the best systematization of the science without them.

I will call such principles "apodictic." Possible examples might be Church's thesis and the (number-theoretic) principle of mathematical induction. Although in the former case, we seem to be unable to prove that recursiveness is effective computability, we might argue that application of the ideal of systematization to transcendent logico-mathematical knowledge shows that the thesis is essential to its best systematization. The introduction of the category of Apodictic Principle does not commit us to formal knowledge's being demonstrably necessary or demonstrably apodictic. As realists, we are committed to there being a fact of the matter in the case of the apodictic truths of formal science, as we are committed to there being one in the case of the necessary truths of formal science, but we are not committed to always being able to know the facts.[12] Realists have no epistemic chutzpah.

To give our conception of transcendent formal knowledge systematicity, I will assume that the category Apodictic Principle encompasses all transcendent formal knowledge that cannot be directly shown to be necessary. As far as I can see, there is no way to establish that formal knowledge can either be directly shown to be necessary on the basis of intuition, some intuitionlike form of reason, or proof, or else be shown to be essential to the best systematization of the body of knowledge in question. But then again, a very similar assumption is made in connection with transcendent natural knowledge. Both seem to derive from the higher-order notion of systematization.

Given this assumption, the question of whether formal knowledge is uniformly *a priori* is the question of whether knowledge of apodictic principles is *a priori*. The answer I want to give is that, in spite of the fact that apodictic principles cannot be directly shown to be necessary, knowledge of them is *a priori* because it is established on the basis of reason alone, on the basis of necessary truths themselves established by reason alone. Since knowledge of basic formal facts is *a priori*, since the step from that knowledge to transcendent knowledge of formal laws and theories is also *a priori*, and since, as a consequence, filling gaps and correcting errors is *a priori* too, formal knowledge always has the *a priori* warrant of pure reason. Systems of *a priori* formal knowledge to which we add apodictic principles thus remain *a priori*. Hence, our rationalism about basic formal knowledge can be extended to transcendent laws and theories.

2.6.3

Even though justification in the formal sciences is *a priori*, propositions in those sciences are revisable in principle. We flatly reject Quine's (1961c, 42–46) equation of apriority with unrevisability (as well as his equation of apriority with analyticity). Of course, the revisability of *a priori* propositions in the formal sciences is something quite different from the revisability of *a posteriori* propositions in the natural sciences. Their revisability is revisability in the light of further pure ratiocination, not revisability in the light of further empirical discoveries. This sharp separation of rational and empirical revisability is a consequence of the fundamental difference between formal and empirical knowledge implicit in the preceding sections (and to be stated explicitly in section 2.8). This difference explains why it isn't possible to argue against the *a priori* nature of knowledge of mathematics just on the grounds of the revisability of mathematical beliefs, as, for example, Kitcher (1983) tries to do. In criticizing Kitcher, Hale (1987, 148) observes that "if revisability is to conflict with apriority, it must be revisability for *empirical* reasons."

12. It might be that an argument showing that a principle is essential to achieving the ideal of systematization in the formal sciences is a kind of transcendental argument for the principle. If so, there has to be an explanation of the nature of such arguments, particularly, of how, given our realism, they differ from Kantian transcendental arguments.

There is, however, a well-known sort of argument that purports to show that mathematics is revisable for empirical reasons because continuing to maintain an alleged *a priori* mathematical truth would be irrational if a better overall empirical theory can be obtained once the proposition is given up. Perhaps the best-known example of this sort of argument is Putnam's (1975b, xv–xvi) claim that abandoning Euclidean geometry in physics is a counterexample to its apriority: "[s]omething literally *inconceivable* had turned out to be true." He (1975b, xv–xvi) writes:

> I was driven to the conclusion that there was such a thing as the overthrow of a proposition that was once *a priori* (or that once had the status of what we *call* "*a priori*" truth). If it could be rational to give up claims as self-evident as the geometrical proposition just mentioned, then, it seemed to me that there was no basis for maintaining that there are *any absolutely a priori truths*, any truths that a rational man is *forbidden* to even doubt.

Putnam's case does not support his conclusion. It is wrong to say that the proposition that was overthrown "was once *a priori*" or "once had the status of what we *call* "*a priori*" truth." To be sure, people once believed that it is *a priori* that Euclidean geometry is a true theory of physical space, and they could not conceive of its not being a true theory of physical space, but Euclidean geometry was never determined *a priori* to be a true theory of physical space. From the fact that it is believed that p is *a priori*, it does not follow that p is *a priori*. In fact, the grounds for accepting Euclidean geometry as a true theory of physical space were straightforwardly *a posteriori*. The abandonment of Euclidean geometry in physics was a revision of an empirical theory. What everyone believed—and what Einstein showed to be false—was a theory in natural science that claimed that the geometric structure of physical space is Euclidean. Putnam's case shows at most that there are no absolutely indubitable propositions in natural science. Since the case concerns an empirical application of Euclidean geometry, the case is one in which an *a posteriori* applied geometry was falsified on empirical grounds, not one in which an *a priori* pure geometry was falsified on such grounds. Thus, the Einsteinian revolution provides no basis for the claim that pure mathematics is in some broad (perhaps Quinean) sense empirical.[13]

Putnam-style arguments to show that mathematics is empirically revisable presuppose Quine's holistic empiricism. Presupposing that scientific knowledge (whether formal or natural) forms a single integrated empirical theory ensures that the disputed *a priori* truth is part of the all-encompassing empirical theory. Without that presupposition the critic of *a priori* knowledge can no longer argue that holding on to a mathematical proposition would be irrational if a better overall empirical theory can be obtained by giving it up. If, instead of constituting one all-encompassing system of empirical beliefs, science were to be divided into two separate domains of belief, one containing formal theories about abstract objects and the other containing empirical theories about concrete objects, the disputed *a priori* mathematical truth in Putnam-style arguments would belong to the former and the empirically refuted proposition would belong to the latter. The disputed *a priori* mathematical truth would thus be a different proposition from any *a posteriori* proposition that could be given up to improve the current empirical theory.

The Quine-Putnam thesis that mathematics is legitimized in virtue of the indispensability of numbers for natural science is a form of what we called methodological naturalism in the introduction. This view, that the only way to have knowledge is through natural science, also presupposes a Quinean empiricism. Without that thesis, there is no reason to think that the legitimacy of mathematical theories about numbers and the like depends solely on their role in natural science. The view thus ignores arguments for their legitimacy based on their indispensability in pure mathematics. If they are indispensable for doing pure mathematics, they are legitimate, and empirical science doesn't have to enter the picture.

As I argued in the last chapter in connection with Field's nominalism, the grounds for acknowledging mathematical objects are not restricted

13. On a realist view, a pure geometry, Euclidean or otherwise, is a theory of a class of abstract spatial structures. In a complete theory, its principles express the possibilities of figures and relations among them within a space. Anything that conflicts with the principles is an impossibility in the space. Grammars, as I (1981) have argued, can be conceived in a similar way, as theories of a class of abstract sentential structures the principles of which express the possibilities of linguistic forms and grammatical relations within a language. In making a place for the notions of necessity and possibility in connection with pure geometries and pure grammars, we can bring geometric and grammatical knowledge under the scope of our rationalist epistemology. In chapter 5, I present an account of the distinction between pure and applied geometries, pure and applied grammars, and so on in terms of the different kinds of objects they study.

to their role in natural science. We could establish their existence even if there were no empirical science. We could do enough mathematics and philosophy of mathematics prior to the development of empirical science to know about the plenitude of numbers and hence to have an argument that numbers cannot be identified with natural objects. It is even conceivable that we could do enough mathematics as disembodied Cartesian beings or in dreams to have a basis for positing numbers, sets, and other mathematical objects.

Since the Quine-Putnam argument for the legitimacy of mathematics rests on Quinean empiricism, it rests on not just a contested empiricism, but, as we shall see in the next chapter, on an incoherent empiricism.

2.7 Have Any Questions Been Begged?

This completes our sketch of a rationalist epistemology to explain why contact with logical and mathematical objects is not necessary for logical and mathematical knowledge. It does not, however, complete our response to Benacerraf's epistemic challenge. To do that, we need to deal with two general doubts that might be raised about whether the epistemology that we have sketched is an adequate response to the challenge.

2.7.1

The first doubt is that our rationalist epistemology might beg the question, since the reasoning it sees as justifying principles in mathematics and logic sometimes rests on the very principles that the reasoning is supposed to justify.[14] Since the acceptability of such reasoning depends on the acceptability of the principles, we would already have had to accept the principles to accept the grounds that the reasoning provides.

This doubt raises a more general doubt, one that goes well beyond the question of our success here. Logicians too are in the situation of attempting to justify inferential principles on the basis of inferences the validity of which depends explicitly on the principles they purport to

14. Circularity would be the wrong way to describe the alleged difficulty. An argument is circular when the very same proposition appearing as the conclusion appears as a premise. On a realist view of mathematical and logical principles, this could never happen. Realism sharply separates the objects of formal knowledge themselves, i.e., numbers, sets, logical principles, proofs, and the like, from our inner epistemic states and processes that provide us with knowledge of them. The former are abstract, objective, and autonomously existing, while the latter are concrete, subjective, and mind-dependent. Thus, the reasoning that provides us with grounds for taking mathematical and logical principles to be true cannot contain those abstract principles.